Enhanced matching perturbation attenuation with discrete-time implementations of sliding-mode controllers

Olivier Huber, Vincent Acary and Bernard Brogliato

***Abstract*— Continuous-time Sliding Mode Control yields when embedded into Filippov's mathematical framework, closed-loop systems with a set-valued controller, represented by differential inclusions. In particular, besides finite-time convergence to the sliding surface and robustness to matched disturbances, such controllers allow an exact compensation of the disturbance on the sliding manifold. In other words, the set-valued input is the exact copy of minus the perturbation. A novel discretization methodology has been recently introduced by the authors, which is based on an implicit discretization of the Filippov's differential inclusion, which in theory totally suppresses the chattering due to the discretization (numerical chattering). In this work we propose an extension of the implicit method, enhancing the perturbation attenuation (in terms of chattering) by using previous values of the set-valued input. This allows to estimate on-line the unknown perturbation, with a time delay due to the sampling. Simulation results illustrate the effectiveness of the method.**

I. Introduction

The time discretization of sliding-mode controllers has witnessed an intense activity in the past 30 years [1], [2], [3], [4], [5], [6] and [7]. This concerns in particular the classical Equivalent-Control-Based Sliding-Mode Control (ECB-SMC), which consists of two sub-controllers: the state-continuous equivalent control u^{eq} and the state-discontinuous control u^s. The chattering phenomenon is seen as a limiting factor to a more widespread deployment of SMC. In this note, from the different sources of chattering, we shall consider the one originating from the discretization of the controller and the matching perturbations. In these past research efforts, most of the focus was on the discontinuous part of the control, since it introduces numerical chattering. In previous works [8], [9] and [10], a new kind of discrete-time controller was proposed. The basic idea is to implement the discontinuous input u^s in an implicit form, while keeping its causality (i.e. the controller is nonanticipative). Then this input has to be computed at each sampling time as the solution to a generalized, set-valued equation, which takes the form of a simple projection on an interval in the simplest cases. The other source of chattering considered here is the perturbations that comprise noise, unmodeled dynamics, and unknown inputs. One of the features of continuous-time sliding-mode control is the disturbance rejection of matched perturbations [11]. However this nice property is not preserved through discretization. Efforts aiming at improving the accuracy with discrete-time controllers include the works in [12] and [13]. In the latter, a constant prediction of the perturbation was used to improve a deadbeat controller. The work in [14], unknown to the authors at the time of writing, shares some similarities with the present one. Other work like the ones cited at the beginning aim at reducing the chattering in general, which includes the numerical one. The specificity of the approach in [12], [13], [14] and the present work is to have a dedicated treatment for the chattering not originating from the discretization.

In this work, we present a modified controller based on the one found in [10], that takes advantage of the previous measured values. Indeed, in this previous work, the convergence of the discretized discontinuous input to the perturbation as the sampling time goes to 0 was studied. This motivates us to use this information in order to improve the attenuation, using finite difference methods to exploit the past informations. After presenting the basic idea, we introduce finite difference formulæand how they can be used to derive prediction of contribution of the perturbation. We also investigate stability and finite-time convergence of the modified controller.

The type of system under study is of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + B\xi(t), \\ u(t) := u^{eq}(t) + u^s(t) + u^p(t), \\ \sigma(t) := Cx(t), \\ u^s(t) \in -\alpha \operatorname{Sgn}(\sigma(x(t))), \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^p, \sigma(t) \in \mathbb{R}^p$, $C \in \mathbb{R}^{p\times n}$, and $\alpha > 0$. The function σ is called the *sliding variable* and the disturbance is denoted as ξ. As one can infer from the dynamics in (1), in this note only matched perturbations are considered. Furthermore we restrict ourselves to the case where ξ is smooth, with no noise component in it. Let us denote by u^p the new control input, designed to cancel the effect of the perturbation. The method used to discretize the dynamics is called Zero-Order Hold (ZOH).

In the remainder of this section, we introduce the notation. In Section II we briefly recall the discrete-time sliding mode controller introduced in [10]. The basic idea is exposed in Section III and the proposed estimation scheme is presented in Section IV. Section V presents the implementation details of the controller. Simulation results using different time-discretization methods are shown in Section VI, to illustrate the different predictions. Finally, stability results are derived in Section VII. Conclusions end the paper in Section VIII.

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The authors are with INRIA Grenoble Rhône-Alpes, BIPOP project-team, 655 avenue de l'Europe, Inovallée, 38334 Saint-Ismier; {vincent.acary,bernard.brogliato}@inria.fr, olivier.huber@inria.fr

Notations: Let $\mathbf{x}\colon \mathbb{R}_+ \times \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$ be the solution of system (1), $x := \mathbf{x}(\,\cdot\,, u, x_0)$ is the solution associated with a continuous-time control u and an initial state $x_0 \in \mathbb{R}^n$, while $\bar{x} := \mathbf{x}(\,\cdot\,, \bar{u}, x_0)$ is the solution with a step function $\bar{u}$ and the same initial state. In the latter case, we denote by $\bar{\sigma} := C\bar{x}$ the sliding variable. The control values change at predefined time instants t_k, defined for all $k \in \mathbb{N}:\ t_k := t_0 + kh,\ t_0, h \in \mathbb{R}_+$. The scalar h is the sampling period. We denote $\bar{x}_k := \bar{x}(t_k)$ and $\bar{\sigma}_k := \bar{\sigma}(t_k)$ for all $k \in \mathbb{N}$. Let $\|\cdot\|$ be the usual Euclidean norm.

Definition 1 (Multivalued sign function)**.** Let $x \in \mathbb{R}$. The multivalued sign function $\mathrm{Sgn}\colon \mathbb{R} \rightrightarrows \mathbb{R}$ is defined as:

$$\mathrm{Sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0. \end{cases} \tag{2}$$

If $x \in \mathbb{R}^n$, then the multivalued sign function $\mathrm{Sgn}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined as: for all $j = 1, \ldots, n$, $(\mathrm{Sgn}(x))_j := \mathrm{Sgn}(x_j)$.

Definition 2. Let $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^p$ and $l \in \mathbb{R}$. One has $f = \mathcal{O}(h^l)$ if for all $x \in \mathbb{R}^n$, there exists $c \in \mathbb{R}^p$ such that $f(x, h)/h^l \to c$ as $h \to 0$.

Definition 3 (Discrete-time sliding phase [10])**.** A system (1), in its sampled-data form, is in the discrete-time sliding phase if $\bar{u}^s$ takes values in $(-\alpha, \alpha)^p$, with α the gain of the controller.

II. Discrete-time controller

We consider $\bar{u}^{eq}$, $\bar{u}^s$ and $\bar{u}^p$ to be right-continuous step functions, with for instance $\bar{u}^s(t) = \bar{u}^s_k$ if $t \in [t_k, t_{k+1})$. Following previous works in [8], [9], [10], $\bar{u}^{eq}$ is designed directly in discrete-time and $\bar{u}^s$ is obtained by an implicit discretization of the Sgn multifunction. Integrating the nominal (that is $\xi \equiv 0$) version of system (1) over $[t_k, t_{k+1})$, we get the ZOH discretization of the system:

$$\bar{x}_{k+1} = e^{Ah}\bar{x}_k + B^*\bar{u}^{eq}_k + B^*\bar{u}^s_k + B^*\bar{u}^p_k, \tag{3}$$

with

$$B^* := \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} B \mathrm{d}\tau. \tag{4}$$

The control inputs are computed as follows: the equivalent part $\bar{u}^{eq}$ is given by the relation

$$CB^*\bar{u}^{eq}_k = C(I - e^{Ah})\bar{x}_k, \tag{5}$$

and with the implicit discretization, for each $k \in \mathbb{N}$, $\bar{u}^s_k$ is the solution to the generalized equation

$$\begin{cases} \widetilde{\sigma}_{k+1} = \bar{\sigma}_k + CB^*\bar{u}^s_k \\ \bar{u}^s_k \in -\alpha\,\mathrm{Sgn}(\widetilde{\sigma}_{k+1}). \end{cases} \tag{6}$$

The computation of the input $\bar{u}^p$ is described in the next section. If we set $u^p \equiv 0$, the closed-loop system (1) is the one studied in [10], and exhibits properties as finite-time duration of the reaching phase and convergence of the discretized discontinuous input $\bar{u}^s$ to the continuous-time one u^s. In Section VII, we shall investigate when those properties are also shared by our modified controller.

III. Problem statement

In continuous time, while in the sliding phase, the discontinuous control input u^s takes values to reject the perturbation action, according to Filippov's concept of solutions. Let us illustrate this using an academic example, $\dot{x}(t) \in -\mathrm{Sgn}(x) + d(t)$, x scalar and d a continuous unknown perturbation, with $|d(t)| \leq 1$ for all t. In the sliding phase, the value taken by $\mathrm{Sgn}(x(t))$ is $d(t)$. In [10], it was established that in the discrete-time sliding phase the input $\bar{u}^s$ is approximating the opposite of contribution of the perturbation in the evolution of the sliding variable. But this action is a posteriori, in the sense that at time t_k, $\bar{u}^s$ corrects for the contribution of the perturbation over the time interval $[t_{k-1}, t_k]$. Hence the attenuation of the perturbation is limited since the control is lagging by one sampling period.

Our approach to enhance the perturbation attenuation consists in using the past values of the sliding variable in order to predict the contribution of the perturbation on the next time interval. Then we take into account this prediction in the control input that is going to be applied for this time interval. As we shall see, this additional control input is computed independently of the other ones.

Let us study the dynamics of the ZOH-discretized discrete-time system with a nonzero perturbation. We suppose the system has been in the discrete-time sliding phase for more than 1 sampling period. Adding the perturbation term to (3), we get

$$\bar{x}_{k+1} = e^{Ah}\bar{x}_k + B^*\bar{u}^{eq}_k + B^*\bar{u}^s_k + B^*\bar{u}^p_k + p_k \tag{7}$$

and with $\bar{u}^{eq}_k$ as in (5), the sliding variable dynamics is

$$\bar{\sigma}_{k+1} = \bar{\sigma}_k + CB^*(\bar{u}^s_k + \bar{u}^p_k) + Cp_k, \tag{8}$$

with

$$p_k := \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} B\xi(\tau)\mathrm{d}\tau. \tag{9}$$

We refer to p_k or Cp_k as the effect, or contribution, of the perturbation. From (6), in the discrete-time sliding phase $\widetilde{\sigma}_{k+1} = 0$, so $\bar{\sigma}_k + CB^*\bar{u}^s_k = 0$. Therefore the relation (8) becomes

$$\bar{\sigma}_{k+1} = Cp_k + CB^*\bar{u}^p_k. \tag{10}$$

If we have an estimate $\widetilde{Cp}_k$ of Cp_k at the time t_k, then we can use this information in the control input u^p, defined as

$$\bar{u}^p_k := -(CB^*)^{-1}\widetilde{Cp}_k. \tag{11}$$

Then injecting (11) in (10), we get:

$$\bar{\sigma}_{k+1} = Cp_k - \widetilde{Cp}_k =: C\hat{p}. \tag{12}$$

In the rest of this note, we name the quantity denoted by $C\hat{p}$ the residual perturbation. The chattering on the output $\bar{\sigma}$ is equal to the prediction error on the effect of the perturbation. Therefore to reduce the chattering imputable to perturbations like unmodeled dynamics, we can try to find a good prediction of the value of the perturbation. The better it is, the smaller the chattering will be. In the next section, we build an estimate $\widetilde{Cp}_k$ of Cp_k at the time t_k.

IV. Prediction of the perturbation

The method we propose to make use of is finite difference, itself based on the Taylor expansion of a function. For instance if we suppose that a function $f\colon \mathbb{R} \to \mathbb{R}^n$ is twice differentiable, it holds that

$$f(x+h) = f(x) + h\dot{f}(x) + \frac{h^2}{2}\ddot{f}(x) + \mathcal{O}(h^3). \tag{13}$$

To use this recurrence formula for the prediction at the next time instant, one need to estimate the values of the first and second derivatives of the function. We use again the finite difference method to get them. We suppose we have access to Cp_k for the previous r sampling periods. However $Cp(t)$ is not a continuous function. It is piecewise smooth, but only right-continuous. From the definition in (9), we have $Cp(t_k^+) = 0$ for all $k \in \mathbb{N}$. This prevents us from directly applying the Taylor expansion to $Cp(t)$ itself. Nonetheless, we can use the Taylor expansion formula on the perturbation ξ and then derive the one for the prediction of Cp_k. Let us illustrate this using a second order expansion of ξ:

$$\xi(\tau+h) = \xi(h) + h\dot{\xi}(\tau) + \frac{h^2}{2}\ddot{\xi}(\tau) + \mathcal{O}(h^3). \tag{14}$$

Multiplying by $e^{A(t_k-\tau)}$ and integrating, we get:

$$\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B\xi(\tau+h)\mathrm{d}\tau = \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B\xi(\tau)\mathrm{d}\tau + \tag{15}$$

$$h\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B\dot{\xi}(\tau)\mathrm{d}\tau + \frac{h^2}{2}\int_{t_{k-1}}^{t_k} e^{A(t_{k+1}-\tau)}B\ddot{\xi}(\tau)\mathrm{d}\tau + \mathcal{O}(h^4). \tag{16}$$

Using the change of variable $\tau' = \tau + h$ in the left-hand side of (16) and using (9), we obtain the relation

$$Cp_k = Cp_{k-1} + hC\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B\dot{\xi}(\tau)\mathrm{d}\tau \tag{17}$$

$$+\frac{h^2}{2}C\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B\ddot{\xi}(\tau)\mathrm{d}\tau + \mathcal{O}(h^4). \tag{18}$$

Let us define $Cp_k^{(i)} := C\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}\xi^{(i)}(\tau)\mathrm{d}\tau$. We use the finite difference method to estimate the second and third terms in (18).

Lemma 1. *Suppose we have an r-step approximation of the i-th derivative of ξ, $\xi^{(i)}(\tau) = \sum_{l=0}^{r}\alpha_l\xi(\tau - lh) + \mathcal{O}(h^{r-i+1})$ with $r \geq i$. Then the approximation formula for $Cp_k^{(i)}$ is $\sum_{l=0}^{r}\alpha_l Cp_{k-l}$ and is of order $\mathcal{O}(h^{r-i+2})$.*

Proof. Starting from the approximation relation for the derivative and with basic operations, we get:

$$C\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}B\xi^{(i)}(\tau)\mathrm{d}\tau = \tag{19}$$

$$\sum_{l=0}^{r}\alpha_l C\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}B\xi(\tau - lh)\mathrm{d}\tau + \mathcal{O}(h^{r-i+2}). \tag{20}$$

For each integral, we use the change of variable $\tau' = \tau - lh$.

$$C\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}B\xi^{(i)}(\tau) = \tag{21}$$

$$\sum_{l=0}^{r}\alpha_l C\int_{t_{k-l}}^{t_{k+1-l}} e^{A(t_{k+1-l}-\tau')}B\xi(\tau')\mathrm{d}\tau' + \mathcal{O}(h^{r-i+2}) \tag{22}$$

$$= \sum_{l=0}^{r}\alpha_l Cp_{k-l} + \mathcal{O}(h^{r-i+2}). \tag{23}$$

□

Using Lemma 1 for the approximate value of the first and second derivatives in (18) yields:

$$Cp_k \approx Cp_{k-1} + h\sum_{l=0}^{r_1}\alpha_l Cp_{k-l} + \frac{h^2}{2}\sum_{l=0}^{r_2}\alpha'_l Cp_{k-l}. \tag{24}$$

where r_1 and r_2 (both $\leq r$) are the order for the prediction of the first and second derivative. Those, with the coefficients α_l and α'_l, are taken from [15]. As Equation (24) illustrates it, the i-th derivative estimate is multiply by h^i. By Lemma 1 we know that the approximation error for the i-th derivative is of order $\mathcal{O}(h^{r-i+2})$. Then the approximation error introduced by the i-th derivative is of order $\mathcal{O}(h^{r+2})$.

Let us present some possible prediction formulæ. Using only the first derivative, we have the relation

$$Cp_k = 2Cp_{k-1} - Cp_{k-2} + \mathcal{O}(h^3) \tag{25}$$

$$= \widetilde{Cp}_k + \mathcal{O}(h^3), \tag{26}$$

where both the approximation of the first derivative and the prediction have the same order. Adding the second derivative yields

$$Cp_k = \frac{5}{2}Cp_{k-1} - 2Cp_{k-2} + \frac{1}{2}Cp_{k-3} + \mathcal{O}(h^3) \tag{27}$$

$$= \widetilde{Cp}_k + \mathcal{O}(h^3). \tag{28}$$

The order is the same as in (26) since we use a 1-step approximation for $Cp_k^{(1)}$, of order 1 which introduces an error of order $\mathcal{O}(h^3)$ in (28). Therefore to achieve an overall error of order $\mathcal{O}(h^4)$, we need an estimate with an order 2 for $Cp_k^{(1)}$, yielding

$$Cp_k = 3Cp_{k-1} - 3Cp_{k-2} + Cp_{k-3} + \mathcal{O}(h^4) \tag{29}$$

$$= \widetilde{Cp}_k + \mathcal{O}(h^4). \tag{30}$$

From now on, let us refer to (26) as a linear or order 1 prediction and to (28) as an order 2 prediction. Both predictions are said to be high-order in contrast with the one given by the relation

$$\widetilde{Cp_k} = Cp_{k-1}, \tag{31}$$

which is said to be constant. Note that the order of the prediction indicates also the number of times the perturbation has to be differentiable for the approximation order to be guaranteed.

V. CONTROLLER IMPLEMENTATION

Let $p \in \mathbb{N}$ be the approximation order. As long as the closed-loop system is not in the discrete-time sliding phase, we set $\bar{u}^p \equiv 0$. Let $k_0 \in \mathbb{N}$ be such that $\widetilde{\sigma}_{k_0} = 0$, that is the system is in the discrete-time sliding phase, and let $k > k_0$. For $k_0 \leq k < k_0 + p$, we let $\bar{u}^p_k = 0$ and we save $\bar{\sigma}_k$, which is Cp_{k-1} according to (10). For $k \geq p + k_0$, we compute the prediction of the perturbation $\widetilde{Cp}_k$ using one of the formula in (26), (30) or (31) and we set $\bar{u}^p_k = (CB)^{-1}\widetilde{Cp}_k$. At this point, we cannot know Cp_{k-1} directly from $\bar{\sigma}_k$ since $\bar{u}^p_k$ changes the dynamics. From (12), at time t_k, Cp_{k-1} is obtained through the relation

$$Cp_{k-1} = \bar{\sigma}_k + \widetilde{Cp}_{k-1}, \tag{32}$$

where $\bar{\sigma}_k$ is measured and $\widetilde{Cp}_{k-1}$ is known from the last estimation.

In [10], the condition $\|Cp_k\| \leq \alpha\beta$ for all k, with β the smallest eigenvalue of $CB^*_s := (CB + (CB^*)^T)/2$, was used to guarantee a finite-time reaching phase and that the system does not exit the discrete-time sliding phase. This inequality condition $\|\widetilde{Cp}_k\| < \alpha\beta$ gives us a "feasibility set" for the prediction $\widetilde{Cp}_k$. If $\widetilde{Cp}_k > \alpha\beta$, then we project the estimate onto the ball of radius $\alpha\beta$ centered at the origin. With this refinement of our algorithm, we make a first step towards some stability property, which is studied in Section VII.

VI. NUMERICAL EXAMPLE

We illustrate the method on a simple 2 dimensional system, taken from [10]:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\bar{u}(t) + B\xi(t) & A = \begin{pmatrix} 0 & 1 \\ 19 & -2 \end{pmatrix}, \\ \sigma = Cx & \\ \bar{u}(t) = \bar{u}^{eq}(t) + \bar{u}^s(t) & B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{cases} \tag{33}$$

The perturbation ξ is a simple sinusoid: $\xi(t) = \sin(4\pi t)$. It was chosen as an example of unmodeled dynamics in a mechanical system. The matrix A has the eigenvalues $\lambda_1 = 3.47$ and $\lambda_2 = -5.47$. The dynamics on the sliding surface is given by $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, which has eigenvalues 0 and -1. The initial state is $(-15, 20)^T$ and the sampling period is 10^{-2}s. We have simulated with 4 different controllers: the classical implicit one, one with the assumption that the perturbation is constant (31), one with the prediction presented in (26) and the last one with the prediction formula in (30). The simulations run for 150 s and were carried out with the SICONOS software package [16][1]. Figures were created using Matplotlib [17]. We present plots of the state variables in Fig. 1, 2 and 3. Then we display the evolution of the discontinuous control $\bar{u}_s$ in Fig. 4 and 5.

In Fig. 1, the state values for the last second are displayed, when each system features some kind of steady-state behaviour. The improvement obtained using the prediction is clearly visible, to the point that we need to zoom to see the chattering with higher-order estimations. In Fig. 2, detail of

[1] `http://siconos.gforge.inria.fr`

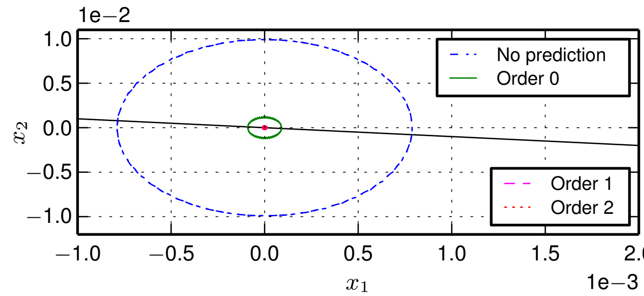

Fig. 1: Simulations of system (33), zoomed around the origin, with $h = 10^{-2}$s.

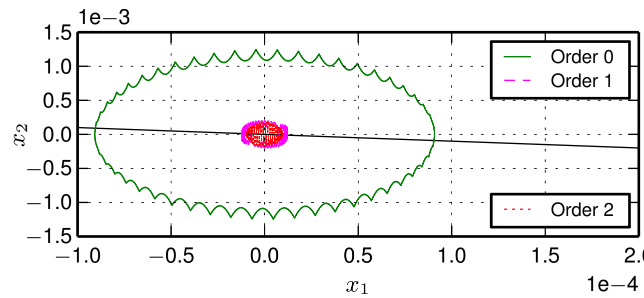

Fig. 2: Detail of Fig. 1, with only the trajectories of the closed-loop systems with prediction visible.

Fig. 1, only the trajectories of the closed-loop systems with prediction are displayed. The behaviour between sampling times can be seen with the constant prediction (order 0). Each high-order predictions (order 1 and 2), yields a closed-loop system featuring even less chattering, as we can see in Fig. 3. At this level of detail, the difference between the

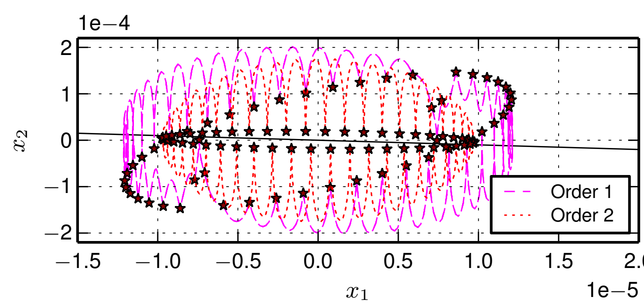

Fig. 3: Detail of Fig. 2, with only the trajectories of the closed-loop system with higher-order prediction visible.

two estimations with high-order becomes visible. Markers indicate the state of the system at each time instant t_k, that is when the control values change. We witness that for those particular values, the prediction with order 2 yields better results with a chattering one order of magnitude smaller. However with the small contribution of the residual perturbation, the inter-sampling dynamics provides an important contribution to the behaviour of the system. For the present example, the advantages of using a prediction order higher than 1 are not so striking as the ones from using at least a linear prediction. Therefore it is important to take into account the inter-sampling dynamics when designing the prediction of the perturbation.

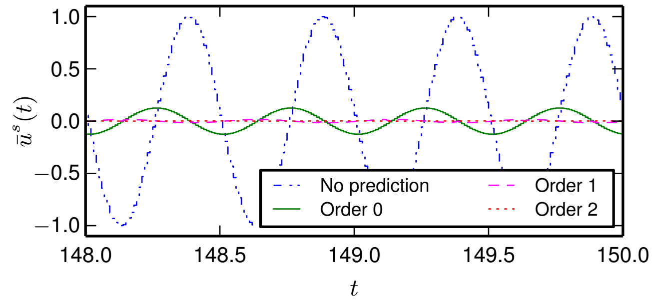

Fig. 4: Evolution of $\bar{u}^s$ for simulations of system (33) with $h = 10^{-2}$s.

Let us turn our attention to the control input. Firstly we plot the evolution of the discontinuous part $\bar{u}^s$, which is, up to a constant, equal to the residual perturbation (12). In Fig. 4, the evolution of $\bar{u}^s$ for the last 2s is displayed. As expected, the better the approximation, the smaller $\bar{u}^s$ is. In

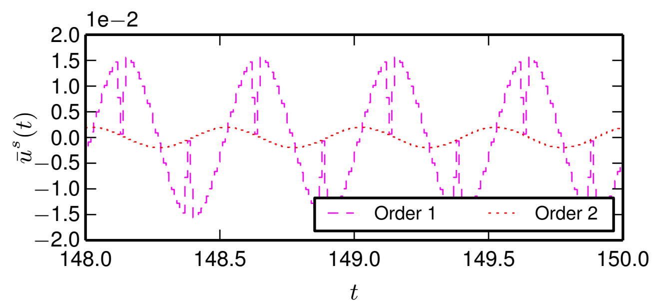

Fig. 5: Detail of Fig. 4.

Fig. 5, detail of Fig. 4, the same phenomenon can be seen for the high-order estimations. Let us turn our attention to $\bar{u}^p$, as

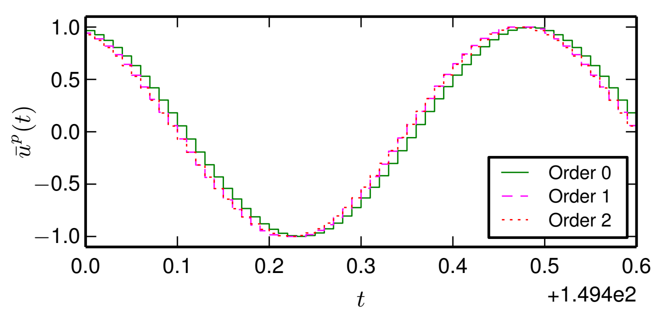

Fig. 6: Evolution of $\bar{u}^p$ for simulations of system (33)

displayed in Fig. 6. From a previous study in [10], we know that with no perturbation prediction, $\bar{u}^s$ approximates the perturbation with a delay of h. Recall that with an order 0, the predicted value of the effect of the perturbation is the one the system just measured. In this case, the values taken by $\bar{u}^p$ are the ones that $\bar{u}^s$ would take with no perturbation prediction. What we can witness from Fig. 6 and 7 is that with a high-order prediction, this lag of one sampling period h seems to have vanished. This is easier to assess on Fig. 7, where if we shift the solid green curve (corresponding with the constant prediction) by $-h$ on the time axis, then it would more or less overlap with the two other curves. This observation can also be explained in the following way: with an estimation of order 0, there is no use of a derivative of ξ to "look into the future". When this is the case, as for the order 1 and 2, the main difference between the different predictions is the value that $\bar{u}^p_k$ takes. This highlights the fact that using high-order estimation yields a prediction, hence a chattering attenuation, substantially better than a constant one. A last

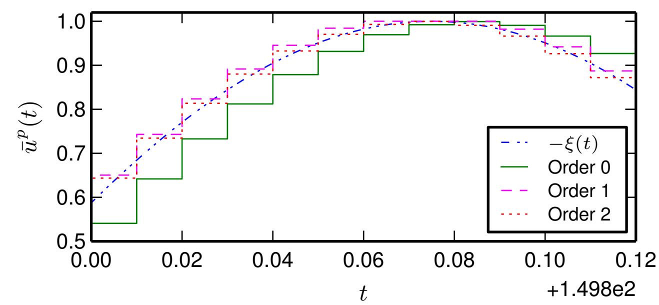

Fig. 7: Detail of Fig. 6.

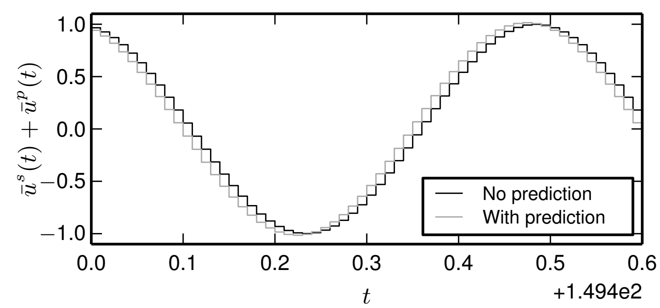

Fig. 8: Evolution of $\bar{u}^s + \bar{u}^p$ for simulations of system (33)

interesting comparison is the control effort used in each of these closed-loop systems. In particular, we want to compare the sum $\bar{u}^s + \bar{u}^p$ with the discontinuous input $\bar{u}^s$ in the case where $u^p \equiv 0$. In Fig. 8, one can see that all the control inputs are quite close, in terms of shape and value. At this level of detail, there is no difference between the closed-loop system with prediction. The main difference is that without prediction, the control is shifted by h with respect to the one with a prediction. Adding a perturbation prediction yields no extra control effort in this example.

VII. Stability properties

One natural question arising from this prediction procedure is whether the use of a bad prediction can degrade the closed-loop performances instead of improving them. Sliding Mode Control is known for its robustness and such a feature should not be lost while trying to reduce the chattering.

First we characterize the robustness with a discrete-time feedback loop. Since we use an implicit discretization of the discontinuous control u^s, we can divide the evolution of the closed-loop system in two phases: the reaching phase, were $\bar{u}^s$ has at least one of its element taking the maximum value $\pm\alpha$ and the discrete-time sliding phase from Definition 3. We define the robustness as the feature that the system does not leave the discrete-time sliding phase, once it enters it.

Proposition 1. *Suppose that CB_s^* is positive-definite and $\beta > 0$ is its smallest eigenvalue. Let the estimate $\widetilde{Cp}_k$ be obtained by the mean of one of the formula in Section IV. If $\alpha > 0$ is such that for all $k \in \mathbb{N}$, $2\|Cp_k\| < \alpha\beta$ and $\widetilde{Cp}_k$ is projected onto the admissible set $\{v \in \mathbb{R}^p \colon \|v\| < \alpha\beta\}$, then the perturbed closed-loop system given by* (8) *and* (6) *enters the discrete-time sliding phase in finite time and stays in it.*

Proof. The proof is similar to one in [10] and is therefore omitted on brevity ground. □

The closed-loop system does not exit the discrete-time sliding surface, even if the estimate is quite far from the actual value. In the worst case the magnitude of the chattering is doubled. The only change change required by the prediction is to double the bound on the discontinuous control input $\bar{u}^s$. This is possible without degrading the performance with the implicit discretization of u^s, since the numerical chattering is non-existent with a LTI system discretized using ZOH.

Remark 1. If it is not possible to double the bound α, it is possible to project the estimate onto a ball with a smaller radius $\gamma\alpha\beta$, where $(1+\gamma)\alpha$ is less than the componentwise upper bound on the control input. It is then easy to adapt the previous proposition to have a result with this estimate. However the chattering reduction might be less appealing.

VIII. Conclusion

In this note, a modified discrete-time sliding mode controller is proposed, which can improved the perturbation attenuation of smooth perturbations. The core idea is to use the previous values of the sliding variable to provide an estimate of the effect of the perturbation on the system for the next sampling period. The method proposed here is adapted from finite-difference formulæ. Numerical simulations obtained with the INRIA software package SICONOS, illustrate the effectiveness of the proposed method. Stability properties of the new controller are also discussed. Experimental comparisons between discrete-time controllers are currently conducted and are about to be submitted for publication. They illustrate that the implicit discretization allows to drastically reduce the output and the input chattering effects, as well as the input magnitude (contrarily to explicit discretizations which invariably lead to bang-bang like controllers with full switching at very high frequency). Further works will include carrying on experimental comparisons as well as dealing with perturbation having a noise component, and also studying the convergence of the proposed controller as $h \to 0$.

References

[1] S. Sarpturk, Y. Istefanopulos, and O. Kaynak, "On the stability of discrete-time sliding mode control systems," *Automatic Control, IEEE Transactions on*, vol. 32, no. 10, pp. 930–932, 1987.

[2] S. Drakunov and V. Utkin, "On discrete-time sliding modes," in *Proc. of* IFAC *Nonlinear Control System Design Conf.*, 1989, pp. 273–278.

[3] K. Furuta, "Sliding mode control of a discrete system," *Systems & Control Letters*, vol. 14, no. 2, pp. 145–152, 1990.

[4] V. Utkin, "Sliding mode control in discrete-time and difference systems," in *Variable Structure and Lyapunov Control*, ser. Lecture Notes in Control and Information Sciences. Springer, 1994, vol. 193, pp. 87–107.

[5] W. Gao, Y. Wang, and A. Homaifa, "Discrete-time variable structure control systems," *Industrial Electronics, IEEE Transactions on*, vol. 42, no. 2, pp. 117–122, 1995.

[6] G. Golo and Č. Milosavljević, "Robust discrete-time chattering free sliding mode control," *Systems & Control Letters*, vol. 41, no. 1, pp. 19–28, 2000.

[7] Č. Milosavljević, "General conditions for the existence of a quasi-sliding mode on the switching hyperplane in discrete variable structure systems," *Automation and Remote Control*, vol. 46, no. 3, pp. 307–314, 1985.

[8] V. Acary and B. Brogliato, "Implicit Euler numerical scheme and chattering-free implementation of sliding mode systems," *Systems & Control Letters*, vol. 59, no. 5, pp. 284–293, 2010.

[9] V. Acary, B. Brogliato, and Y. Orlov, "Chattering-free digital sliding-mode control with state observer and disturbance rejection," *Automatic Control, IEEE Transactions on*, vol. 57, no. 5, pp. 1087–1101, 2012.

[10] O. Huber, V. Acary, and B. Brogliato, "Analysis of explicit and implicit discrete-time equivalent-control based sliding mode controllers," INRIA, Rapport de Recherche RR-8383, 2013. [Online]. Available: http://hal.inria.fr/hal-00875209

[11] B. Drazenovic, "The invariance conditions in variable structure systems," *Automatica*, vol. 5, no. 3, pp. 287–295, 1969.

[12] D. Mitic and Č. Milosavljević, "Sliding mode-based minimum variance and generalized minimum variance controls with $\mathcal{O}(T^2)$ and $\mathcal{O}(T^3)$ accuracy," *Electrical Engineering (Archiv fur Elektrotechnik)*, vol. 86, no. 4, pp. 229–237, 2004.

[13] W.-C. Su, S. Drakunov, and Ü. Özgüner, "An $O(T^2)$ boundary layer in sliding mode for sampled-data systems," *Automatic Control, IEEE Transactions on*, vol. 45, no. 3, pp. 482–485, 2000.

[14] M. Ljesnjanin, B. Drazenovic, Č. Milosavljević, and B. Veselic, "Disturbance compensation in digital sliding mode," in *EUROCON-International Conference on Computer as a Tool (EUROCON), 2011 IEEE*. IEEE, 2011, pp. 1–4.

[15] B. Fornberg, "Generation of finite difference formulas on arbitrarily spaced grids," *Mathematics of Computation*, vol. 51, no. 184, pp. 699–706, 1988.

[16] V. Acary and F. Pérignon, "An introduction to SICONOS," INRIA, Rapport Technique RT-0340, 2007. [Online]. Available: http://hal.inria.fr/inria-00162911

[17] J. D. Hunter, "Matplotlib: a 2D graphics environment," *Computing in Science & Engineering*, pp. 90–95, 2007.